Dynamics of Vorticity Near the Position of its Maximum Modulus

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7 May 2012

Motivation

- Extreme events in realistic fluids: fields such as vorticity become intense and localised in space and time
- Finite-time singularity problem in ideal fluids
- One would like to understand how vorticity behaves near its maximum
- Does the position of the peak vorticity move with the flow? NO
- How is the spatial structure of vorticity near the peak vorticity?

Outline

3D Navier-Stokes fluid equations

3D Navier-Stokes

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p + \nu \triangle \boldsymbol{u}, \quad (1)$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad (2)$$

where $\boldsymbol{u} \equiv \boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity vector field (assumed smooth), $\boldsymbol{x} \in \mathbb{R}^3$, $t \in [0, T_*)$, and $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$ is the Lagrangian derivative.

Vorticity vector field $\boldsymbol{\omega} \equiv \nabla \times \boldsymbol{u}$ satisfies:

$$\frac{D\omega}{Dt} = (\nabla \boldsymbol{u})^{\mathrm{T}} \omega + \nu \triangle \omega, \quad (3)$$

where $((\nabla \boldsymbol{u})^{\mathrm{T}} \omega)_j = \frac{\partial u_j}{\partial x_k} \omega_k$, $j = 1, 2, 3$, in Cartesian coordinates (Einstein convention over repeated indices).

3D Navier-Stokes fluid equations

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Vorticity modulus $|\omega|$ (1/3)

$$\frac{D\boldsymbol{\omega}}{Dt} = (\nabla \boldsymbol{u})^{\mathrm{T}} \boldsymbol{\omega} + \nu \triangle \boldsymbol{\omega} \qquad \text{(Vorticity Equation)}$$

Vorticity decomposition into modulus and direction:

$$\boldsymbol{\omega} = \omega \boldsymbol{\xi}, \qquad \omega \equiv |\boldsymbol{\omega}|, \qquad |\boldsymbol{\xi}| \equiv 1.$$

- Take the vorticity equation and evaluate the scalar product of each term with the vorticity vector field $\boldsymbol{\omega}$. We get:

$$\begin{aligned} \boldsymbol{\omega} \cdot \frac{D\boldsymbol{\omega}}{Dt} = \omega \frac{D\omega}{Dt} &= \boldsymbol{\omega} \cdot \left((\nabla \boldsymbol{u})^{\mathrm{T}} \boldsymbol{\omega} + \nu \triangle \boldsymbol{\omega} \right), \\ &= \omega^2 \boldsymbol{\xi} \cdot (\nabla \boldsymbol{u}) \boldsymbol{\xi} + \nu \boldsymbol{\omega} \cdot \triangle \boldsymbol{\omega}. \end{aligned}$$

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Vorticity modulus $|\boldsymbol{\omega}|$ (2/3)

$$\boxed{\omega \frac{D\omega}{Dt} = \omega^2 \boldsymbol{\xi} \cdot (\nabla \boldsymbol{u}) \boldsymbol{\xi} + \nu\, \boldsymbol{\omega} \cdot \triangle \boldsymbol{\omega}}$$

- A simple calculation yields

$$\boldsymbol{\omega} \cdot \triangle \boldsymbol{\omega} = -\omega^2 |\nabla \boldsymbol{\xi}|^2 + \omega \triangle \omega\,,$$

so we get

$$\frac{D\omega}{Dt} = \omega\, \boldsymbol{\xi} \cdot (\nabla \boldsymbol{u}) \boldsymbol{\xi} + \nu\, \triangle \omega - \nu\, \omega\, |\nabla \boldsymbol{\xi}|^2 .$$

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Vorticity modulus $|\omega|$ (3/3)

$$\boxed{\frac{D\omega}{Dt} = \omega\,\boldsymbol{\xi}\cdot(\nabla\boldsymbol{u})\boldsymbol{\xi} + \nu\,\Delta\omega - \nu\,\omega\,|\nabla\boldsymbol{\xi}|^2}$$

- Now, defining the effective stretching rate α as:

$$\alpha \equiv \boldsymbol{\xi}\cdot(\nabla\boldsymbol{u})\boldsymbol{\xi} + \nu\,\frac{\Delta\omega}{\omega} - \nu\,|\nabla\boldsymbol{\xi}|^2\,,$$

we arrive at the Constantin-type evolution equation for the vorticity modulus:

$$\frac{D\omega}{Dt} = \omega\,\alpha\,.$$

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Constantin's equation and position of maximum vorticity modulus (1/2)

Constantin's equation (explicit form)

$$\frac{\partial \omega}{\partial t}(\boldsymbol{x},t)+\boldsymbol{u}(\boldsymbol{x},t)\cdot\nabla\omega(\boldsymbol{x},t)=\omega(\boldsymbol{x},t)\,\alpha(\boldsymbol{x},t)\,,\quad \forall\,\boldsymbol{x}\in\mathbb{R}^3\,,\quad \forall\,t\in[0,T_*)$$

- Define the position of a local maximum of vorticity modulus $\omega(\boldsymbol{x},t)$ as the time-dependent vector $\boldsymbol{Y}(t)$ such that:

$$\nabla\omega(\boldsymbol{Y}(t),t)=\boldsymbol{0}\,,\quad \text{with}\quad \frac{\partial^2\omega}{\partial x_j\partial x_k}(\boldsymbol{Y}(t),t)\quad \text{negative-definite.}$$

- Evaluate Constantin's equation at $\boldsymbol{x}=\boldsymbol{Y}(t)$. The gradient term $\nabla\omega(\boldsymbol{Y}(t),t)$ vanishes by definition and we get

$$\frac{\partial \omega}{\partial t}(\boldsymbol{Y}(t),t)=\omega(\boldsymbol{Y}(t),t)\,\alpha(\boldsymbol{Y}(t),t)\,,\quad \forall\,t\in[0,T_*)\,.$$

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$$\boxed{\frac{\partial \omega}{\partial t}(\boldsymbol{Y}(t),t) = \omega(\boldsymbol{Y}(t),t)\,\alpha(\boldsymbol{Y}(t),t), \quad \forall\, t \in [0, T_*)}$$

- Notice now that

$$\frac{d}{dt}[\omega(\boldsymbol{Y}(t),t)] = \frac{\partial \omega}{\partial t}(\boldsymbol{Y}(t),t) + \frac{d\boldsymbol{Y}}{dt}\cdot\nabla\omega(\boldsymbol{Y}(t),t) = \frac{\partial \omega}{\partial t}(\boldsymbol{Y}(t),t).$$

  Comparing this with the boxed equation gives finally:

$$\frac{d}{dt}[\omega(\boldsymbol{Y}(t),t)] = \omega(\boldsymbol{Y}(t),t)\,\alpha(\boldsymbol{Y}(t),t), \quad \forall\, t \in [0, T_*).$$

- Up to here, it is not obvious whether or not $\boldsymbol{Y}(t)$ follows the material particles (but it doesn't).

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Constantin's equations: Test of numerical data (1/3)

$$\boxed{\frac{d}{dt}[\omega(\boldsymbol{Y}(t),t)] = \omega(\boldsymbol{Y}(t),t)\,\alpha(\boldsymbol{Y}(t),t), \quad \forall\, t \in [0, T_*)}$$

- Choose $\boldsymbol{Y}(t)$ to be the position of the global maximum of vorticity modulus, so $\omega(\boldsymbol{Y}(t),t) = \|\boldsymbol{\omega}(\cdot,t)\|_\infty$ (max norm).
- We investigate this max norm using data from a $1024 \times 256 \times 2048$ pseudo-spectral numerical simulation of 3D Euler anti-parallel vortices (Bustamante&Kerr 2007).

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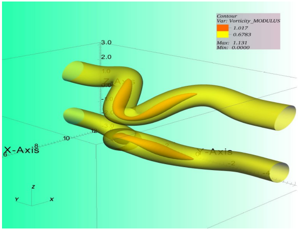
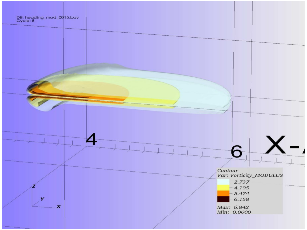
Constantin's equations: Test of numerical data (2/3)

- The position $\boldsymbol{Y}(t)$ is trapped on the "symmetry plane".





- We have stored spatial field data at the symmetry plane, at selected times t between 5.9 and 9.4.
- At each selected time t, a spline spatial interpolation is done to obtain accurate values of the position of vorticity maximum $\boldsymbol{Y}(t)$.

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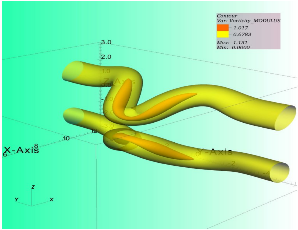

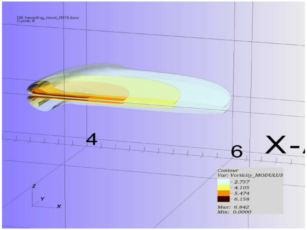

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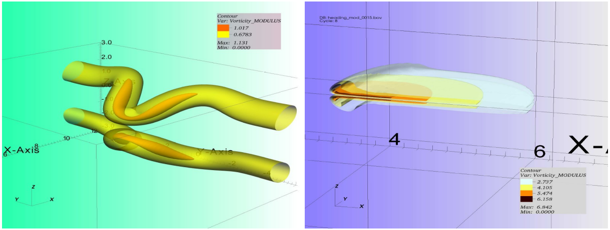

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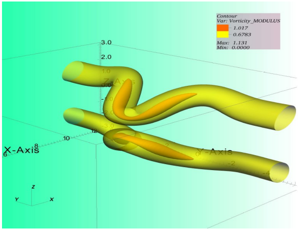

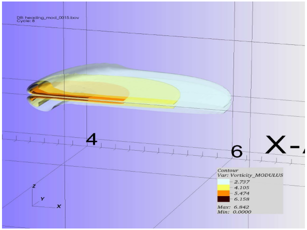

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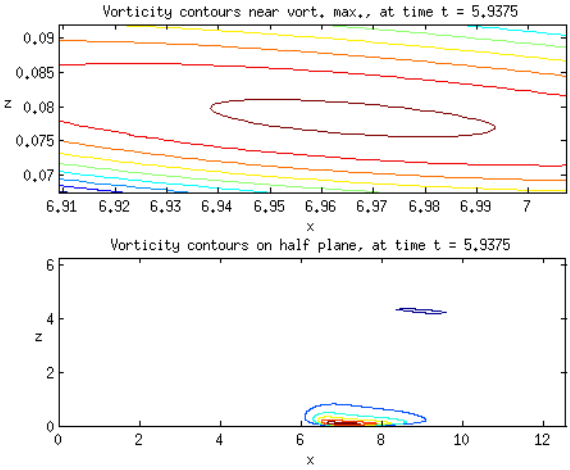
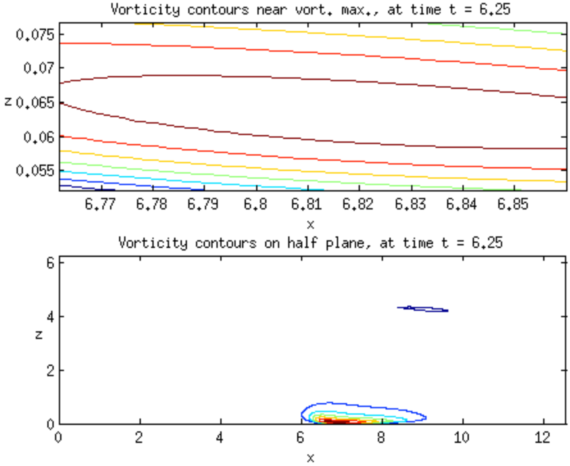
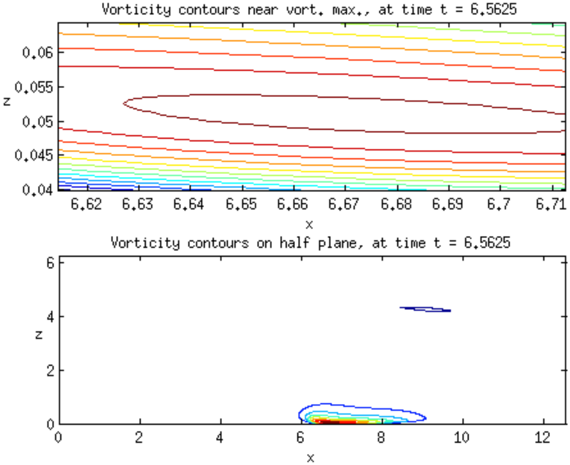
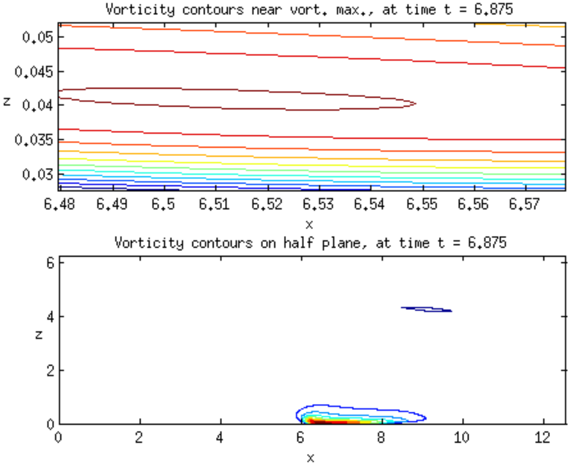
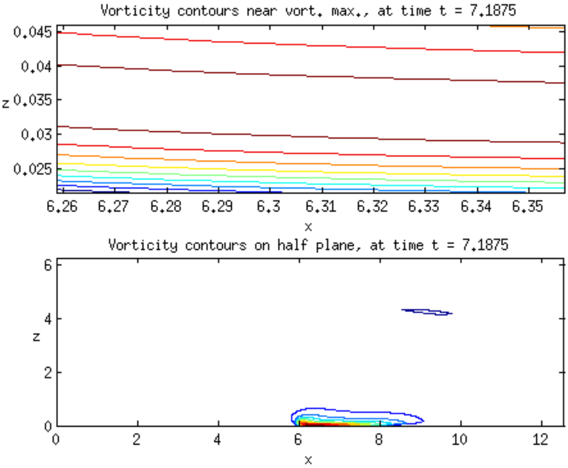
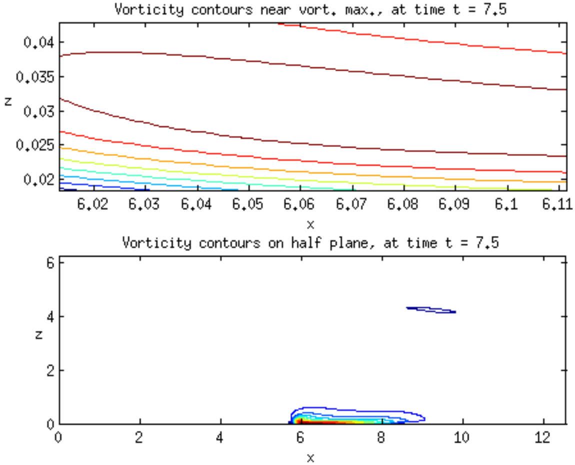

Vorticity contours near vort. max., at time t = 5.9375
0.09
0.085
0.08
0.075
0.07
z
6.91
6.92
6.93
6.94
6.95
6.96
6.97
6.98
6.99
7
x
Vorticity contours on half plane, at time t = 5.9375
6
4
2
0
z
0
2
4
6
8
10
12
x

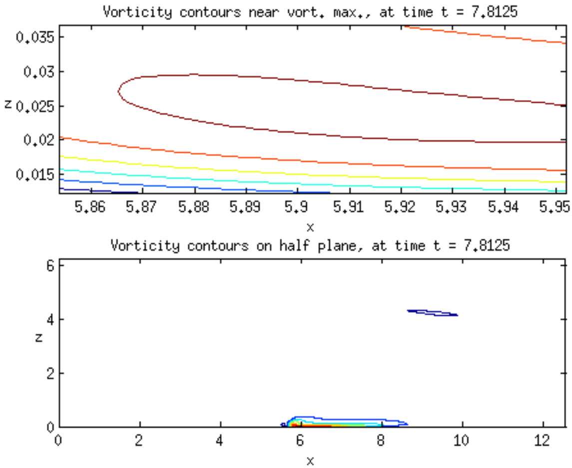

Vorticity contours near vort. max., at time t = 6.25
0.075
0.07
z 0.065
0.06
0.055
6.77
6.78
6.79
6.8
6.81
6.82
6.83
6.84
6.85
x
Vorticity contours on half plane, at time t = 6.25
6
4
z
2
0
0
2
4
6
8
10
12
x

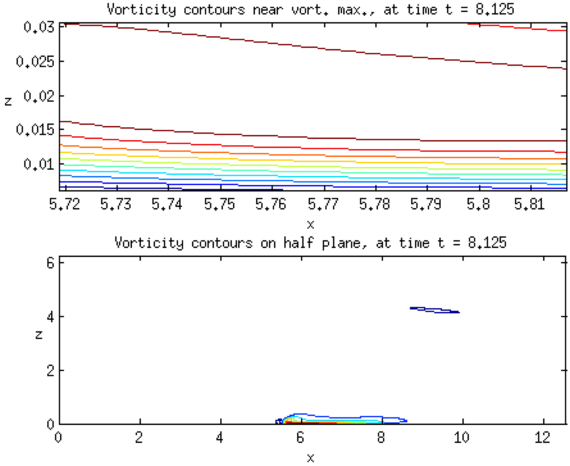

Vorticity contours near vort. max., at time t = 6.5625
0.06
0.055
0.05
0.045
0.04
z
6.62
6.63
6.64
6.65
6.66
6.67
6.68
6.69
6.7
6.71
x
Vorticity contours on half plane, at time t = 6.5625
6
4
2
0
z
0
2
4
6
8
10
12
x

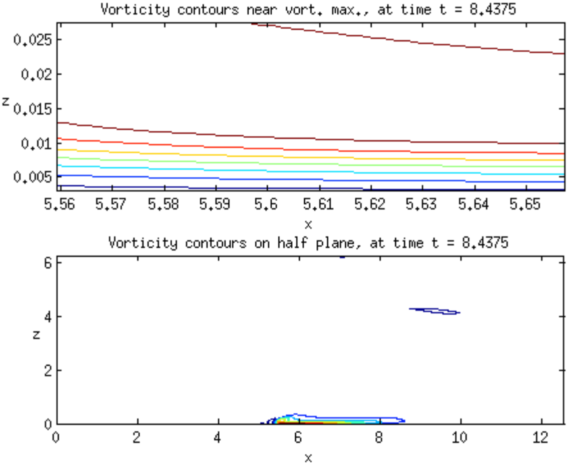

Vorticity contours near vort. max., at time t = 6.875
0.05
0.045
z
0.04
0.035
0.03
6.48
6.49
6.5
6.51
6.52
6.53
6.54
6.55
6.56
6.57
x
Vorticity contours on half plane, at time t = 6.875
6
4
z
2
0
0
2
4
6
8
10
12
x

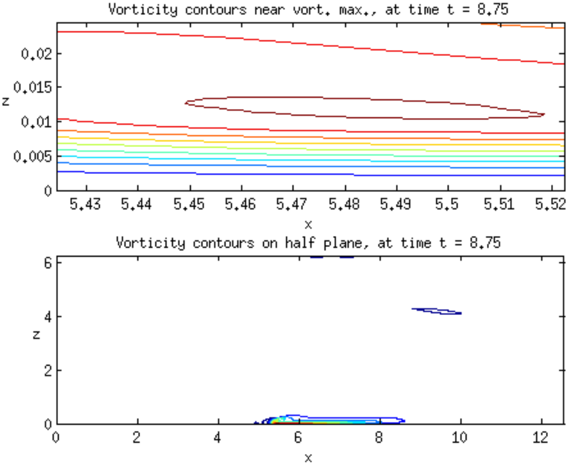

Vorticity contours near vort. max., at time t = 7.1875
0.045
0.04
z 0.035
0.03
0.025
6.26
6.27
6.28
6.29
6.3
6.31
6.32
6.33
6.34
6.35
x
Vorticity contours on half plane, at time t = 7.1875
6
4
z
2
0
0
2
4
6
8
10
12
x

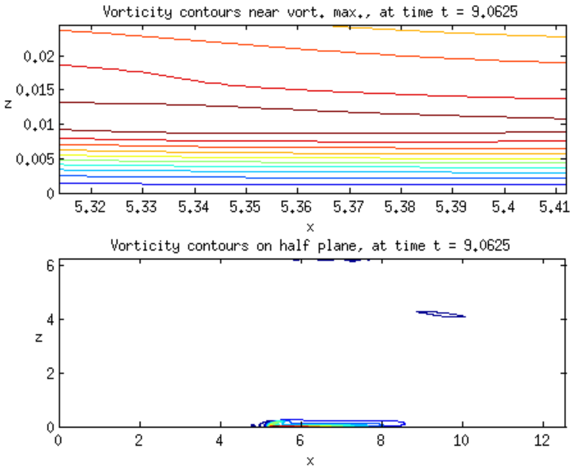

Vorticity contours near vort. max., at time t = 7.5
0.04
0.035
0.03
0.025
0.02
z
6.02
6.03
6.04
6.05
6.06
6.07
6.08
6.09
6.1
6.11
x
Vorticity contours on half plane, at time t = 7.5
6
4
2
0
z
0
2
4
6
8
10
12
x

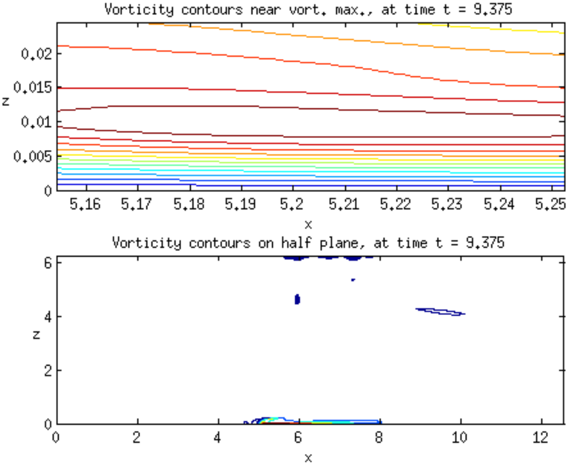

Vorticity contours near vort. max., at time t = 7.8125
0.035
0.03
z 0.025
0.02
0.015
5.86 5.87 5.88 5.89 5.9 5.91 5.92 5.93 5.94 5.95
x
Vorticity contours on half plane, at time t = 7.8125
6
4
z
2
0
0 2 4 6 8 10 12
x

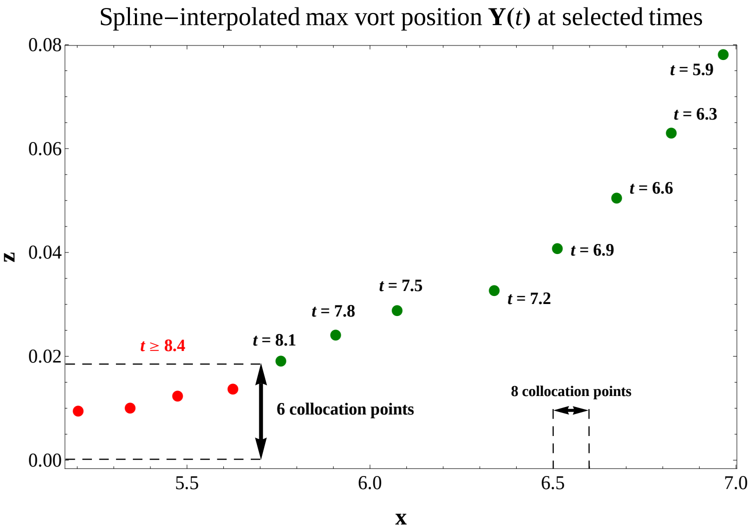

Vorticity contours near vort. max., at time t = 8.125
0.03
0.025
0.02
0.015
0.01
z
5.72
5.73
5.74
5.75
5.76
5.77
5.78
5.79
5.8
5.81
x
Vorticity contours on half plane, at time t = 8.125
6
4
2
0
z
0
2
4
6
8
10
12
x


Vorticity contours near vort. max., at time t = 8.4375
0.025
0.02
z
0.015
0.01
0.005
5.56
5.57
5.58
5.59
5.6
5.61
5.62
5.63
5.64
5.65
x
Vorticity contours on half plane, at time t = 8.4375
6
4
z
2
0
0
2
4
6
8
10
12
x


Vorticity contours near vort. max., at time t = 8.75
0.02
0.015
0.01
0.005
0
z
5.43
5.44
5.45
5.46
5.47
5.48
5.49
5.5
5.51
5.52
x
Vorticity contours on half plane, at time t = 8.75
6
4
2
0
z
0
2
4
6
8
10
12
x


Vorticity contours near vort. max., at time t = 9.0625
0.02
0.015
0.01
0.005
0
z
5.32
5.33
5.34
5.35
5.36
5.37
5.38
5.39
5.4
5.41
x
Vorticity contours on half plane, at time t = 9.0625
6
4
2
0
z
0
2
4
6
8
10
12
x

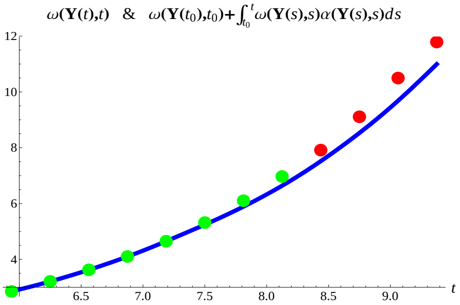

Vorticity contours near vort. max., at time t = 9.375
0.02
0.015
z
0.01
0.005
0
5.16
5.17
5.18
5.19
5.2
5.21
5.22
5.23
5.24
5.25
x
Vorticity contours on half plane, at time t = 9.375
6
4
z
2
0
0
2
4
6
8
10
12
x


Spline−interpolated max vort position Y(t) at selected times
0.08
0.06
0.04
0.02
0.00
z
5.5
6.0
6.5
7.0
x
t = 5.9
t = 6.3
t = 6.6
t = 6.9
t = 7.2
t = 7.5
t = 7.8
t = 8.1
t ≥ 8.4
6 collocation points
8 collocation points

Constantin's equations: Test of numerical data (3/3)

$$\frac{d}{dt}[\omega(\boldsymbol{Y}(t),t)] = \omega(\boldsymbol{Y}(t),t)\,\alpha(\boldsymbol{Y}(t),t), \quad \forall\, t \in [0, T_*)$$

We test the data by evaluating independently the values of $\omega(\boldsymbol{Y}(t),t)$ (green and red bullets), and the time integral of the time-interpolated product $\omega(\boldsymbol{Y}(t),t)\alpha(\boldsymbol{Y}(t),t)$ (blue curve).



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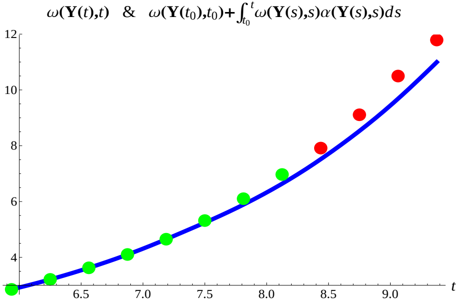

Outline

Evolution of position of maximum vorticity $\boldsymbol{Y}(t)$ (1/2)

By definition: $$\frac{\partial\omega}{\partial x_j}(\boldsymbol{Y}(t),t)=0\,,\quad \forall\, t\in[0,T_*)\,,\quad j=1,2,3.$$

- Take time derivative of the above equation. We get:

$$\frac{d}{dt}\left[\frac{\partial\omega}{\partial x_j}(\boldsymbol{Y}(t),t)\right]=0=\frac{\partial^2\omega}{\partial t\partial x_j}(\boldsymbol{Y}(t),t)+\frac{d\boldsymbol{Y}}{dt}\cdot\frac{\partial\nabla\omega}{\partial x_j}(\boldsymbol{Y}(t),t).$$

- The first term in the RHS of this equation can be simplified using Constantin's equation. We have in general:

$$\begin{aligned}\frac{\partial^2\omega}{\partial t\partial x_j}(\boldsymbol{x},t) &= -\boldsymbol{u}(\boldsymbol{x},t)\cdot\frac{\partial\nabla\omega}{\partial x_j}(\boldsymbol{x},t)-\frac{\partial\boldsymbol{u}}{\partial x_j}\cdot\nabla\omega(\boldsymbol{x},t)\\ &+\frac{\partial\omega}{\partial x_j}(\boldsymbol{x},t)\,\alpha(\boldsymbol{x},t)+\omega(\boldsymbol{x},t)\frac{\partial\alpha}{\partial x_j}(\boldsymbol{x},t).\end{aligned}$$

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Evolution of position of maximum vorticity $\boldsymbol{Y}(t)$ (2/2)

Evaluating this at $\boldsymbol{x} = \boldsymbol{Y}(t)$ we conclude:

$$0 = \left[\frac{d\boldsymbol{Y}}{dt} - \boldsymbol{u}(\boldsymbol{Y}(t), t) \right] \cdot \frac{\partial \nabla \omega}{\partial x_j}(\boldsymbol{Y}(t), t) + \omega(\boldsymbol{Y}(t), t) \frac{\partial \alpha}{\partial x_j}(\boldsymbol{Y}(t), t)$$

so, in terms of the matrix of 2$^{\mathrm{nd}}$ derivatives (i.e., Hessian) of ω,

$$D^2\omega(\boldsymbol{x}, t) \equiv \left[\frac{\partial^2 \omega}{\partial x_j \partial x_k} \right](\boldsymbol{x}, t),$$

which is by definition negative-definite at $\boldsymbol{x} = \boldsymbol{Y}(t)$ and therefore invertible there, we get the "drift" equation:

$$\boxed{\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{u}(\boldsymbol{Y}(t), t) + \omega(\boldsymbol{Y}(t), t) \left[-D^2\omega(\boldsymbol{Y}(t), t) \right]^{-1} \nabla \alpha(\boldsymbol{Y}(t), t).}$$

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Drift equation

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So the position of the global maximum of vorticity does not follow the material particles.
We define the "drift vector field" $\mathfrak{D}(\boldsymbol{x},t)$ for $\boldsymbol{x}$ near $\boldsymbol{Y}(t)$:

$$\mathfrak{D}(\boldsymbol{x},t) \equiv \omega(\boldsymbol{x},t)\left[-D^2\omega(\boldsymbol{x},t)\right]^{-1}\nabla\alpha(\boldsymbol{x},t).$$

Therefore the Drift equation is simply

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Drift equation: Test of numerical data: *x*-coordinate

$$\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{u}(\boldsymbol{Y}(t),t) + \mathfrak{D}(\boldsymbol{Y}(t),t),$$

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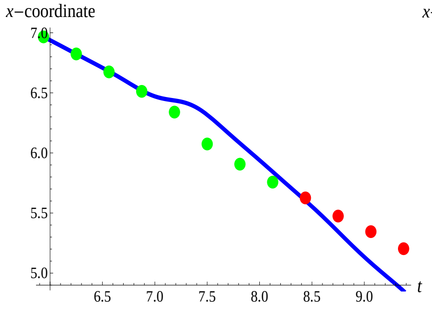
$\mathbf{Y}(t)$ & $\mathbf{Y}(t_0)+\int_{t_0}^{t}\{\boldsymbol{u}(\mathbf{Y}(s),s)+\mathfrak{D}(\mathbf{Y}(s),s)\}\mathbf{ds}$



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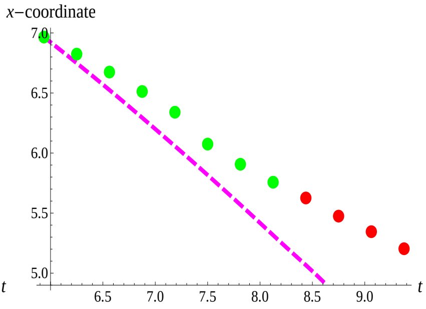
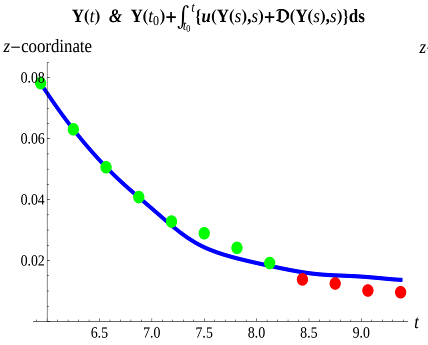
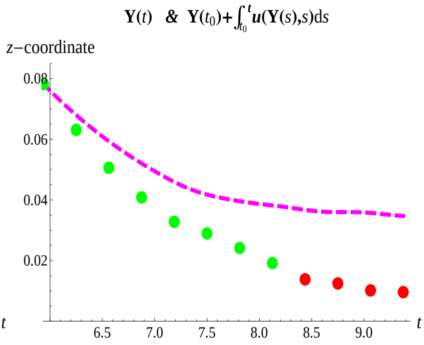


Drift equation: Test of numerical data: *z*-coordinate

$$\frac{d\boldsymbol{Y}}{dt} = \boldsymbol{u}(\boldsymbol{Y}(t),t) + \mathfrak{D}(\boldsymbol{Y}(t),t),$$

$$\mathfrak{D}(\boldsymbol{x},t) = \omega(\boldsymbol{x},t)\left[-D^2\omega(\boldsymbol{x},t)\right]^{-1}\nabla\alpha(\boldsymbol{x},t).$$

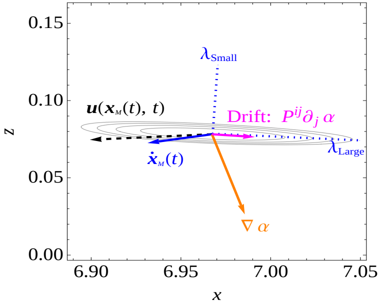




Understanding the drift

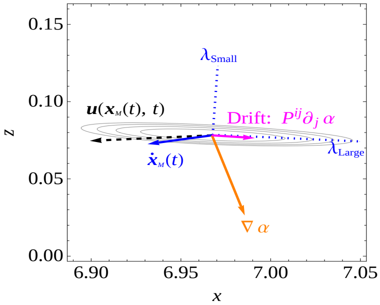
$$\mathfrak{D}(\boldsymbol{x},t) \quad = \quad \omega(\boldsymbol{x},t)\left[-D^2\omega(\boldsymbol{x},t)\right]^{-1}\nabla\alpha(\boldsymbol{x},t)$$

The drift vector points *more or less* in the direction of $\nabla\alpha(\boldsymbol{Y}(t),t)$, but this depends on the local profile of vorticity modulus near the maximum. See $t = 5.9$ snapshot:



$$\mathfrak{D}(\boldsymbol{x},t) \quad = \quad \omega(\boldsymbol{x},t)\left[-D^2\omega(\boldsymbol{x},t)\right]^{-1}\nabla\alpha(\boldsymbol{x},t).$$

Key quantities: eigenvalues of $\omega(\boldsymbol{Y}(t),t)\left[-D^2\omega(\boldsymbol{Y}(t),t)\right]^{-1}$. Their square roots define three independent length scales, $\lambda_1(t), \lambda_2(t), \lambda_3(t)$. Interpretation: as radii of the “nominal” ellipsoids of half-peak vorticity isosurfaces.



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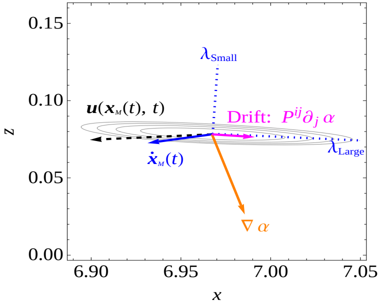

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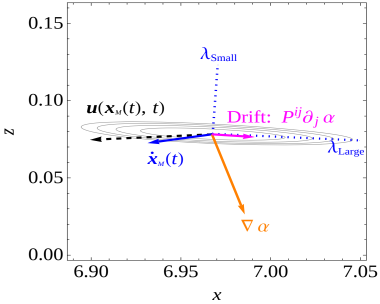

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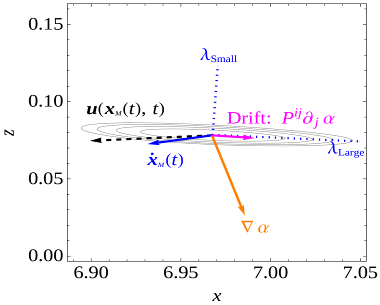
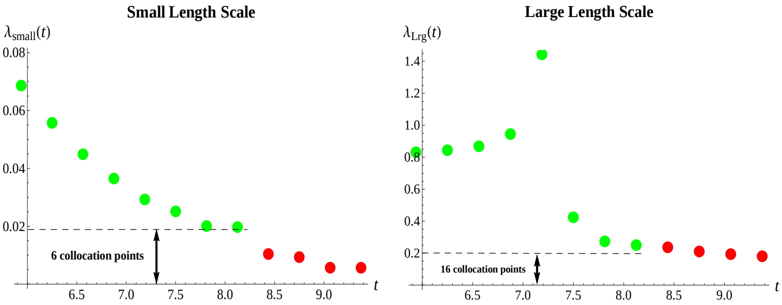


Outline

Direct study from numerical data

Direct computation of eigenvalues of matrix $\sqrt{\omega(\boldsymbol{Y}(t),t)\,[-D^2\omega(\boldsymbol{Y}(t),t)]^{-1}}$ at each selected time, gives the following symmetry-plane length scales:



Equations of motion for length scales

Each of the three length scales satisfies an equation of motion. We state these without proof:

$$\frac{d\lambda_a}{dt} = \lambda_a \boldsymbol{v}_a \cdot \left[(\nabla \boldsymbol{u}) + \frac{1}{2}(\nabla \mathfrak{D}) \right] \boldsymbol{v}_a \,, \quad a = 1,2,3,$$

where $\boldsymbol{v}_a$ are the normalised eigenvectors of $[D^2\omega(\boldsymbol{Y}(t), t)]$.

Application: it is possible to determine how much does the vorticity profile deviate from self-similarity. Self-similar collapse at the symmetry plane would imply that the "vortex blob" has constant circulation:

$$C(t) \equiv \lambda_{\rm small}(t)\, \lambda_{\rm Large}(t)\, \|\omega(\cdot, t)\|_\infty = {\rm const.}$$

Instead, we have, rigorously: $\boxed{\frac{d}{dt} \ln C(t) = \frac{1}{2} \nabla_{\rm 2D} \cdot \mathfrak{D}(\boldsymbol{Y}(t), t)}$

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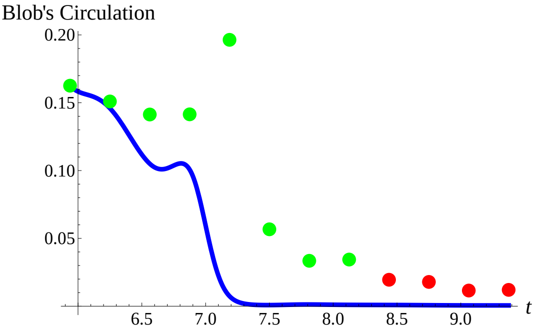
Vortex blob's circulation

$$\frac{d}{dt} \ln C(t) = \frac{1}{2} \nabla_{2D} \cdot \mathfrak{D}(\boldsymbol{Y}(t), t)$$

$$C(t) \;\; \& \;\; C(t_0) e^{\frac{1}{2} \int_{t_0}^{t} \nabla_{2D} \cdot \mathfrak{D}(\boldsymbol{Y}(s), s)\, ds}$$



Conclusions

- We have revealed the laws of motion of the position of the vorticity maximum in 3D Navier-Stokes and Euler
- Fundamental role of new "Drift" vector field
- These laws have been used to check validity of high-resolution numerical simulations
- Fundamental role of the length scales of the vorticity profile near the maximum
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Thank you

Thank you for your attention!